CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar, Friday conversations:,
For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks.

The goal of these talks is threefold

1. Category theory aimed at “free formulas” and their combinatorics
2. How to construct free objects
   1. w.r.t. a functor with - at least - two combinatorial applications:
      1. the two routes to reach the free algebra
      2. alphabets interpolating between commutative and non commutative worlds
   2. without functor: sums, tensor and free products
   3. w.r.t. a diagram: colimits
3. Representation theory.
4. MRS factorisation: A local system of coordinates for Hausdorff groups and fine tuning between analysis and algebra.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

Disclaimers.

Disclaimer I.– The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

Disclaimer II.– Sometimes, absolute rigour is not followed[a]. In its place, from time to time, we will seek to give the reader an intuitive feel for what the concepts of category theory are and how they relate to our ongoing research within CIP, CAP and CCRT.

[a] All is assumed to be subsequently clarified on request though.

Disclaimer III.– The reader will find repetitions and reprises from the preceding CCRT[n], they correspond to some points which were skipped or uncompletely treated during preceding seminars.

The Way We Live Now[2] (after T. Gowers et al.'s introduction).

Bertrand Russell, in his book The Principles of Mathematics, proposes the following as a definition of pure mathematics.

Pure Mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

Russell's book was published in 1903, and many mathematicians at that time were preoccupied with the logical foundations of the subject. It could be said that <u>modern Mathematics is about everything</u> that Russell's definition leaves out[1].

[1] Not disregarding what Russell and his forefathers taught us: logical dependence, soundness of proofs, axiomatic method.

[2] Satirical novel by Anthony Trollope, published in London in 1875.

Wikipedia says

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces .../...
The success of representation theory has led to numerous generalizations. One of the most general is in category theory.

As our track is based on Combinatorial Physics and Experimental/Computational Mathematics, we will have a practical approach of the three main points of view

- Algebraic
- Geometric
- Combinatorial
- Categorical

Matters

1. Representation theory (or theories)
   1. Geometric point of view
   2. Combinatorial point of view (Ram and Barcelo manifesto)
   3. Categorical point of view
2. From groups to algebras
   Here is a bit of rep. theory of the symmetric group, deformations, idempotents
3. Irreducible and indecomposable modules
4. Characters, central functions and shifts.
   Here are (some of) Lascoux and Schützenberger's results
5. Reductibility and invariant inner products
   Here stands Joseph's result
6. Commutative characters
   Here are time-ordered exponentials, iterated integrals, evolution equations and Minh's results
7. Lie groups Cartan theorem
   Here is BTT

CCRT[29] A theory of Domains for Hyper- and Polylogarithms.

Plan.

[a] Satirical novel by Anthony Trollope, published in London in 1875.

Goal of this talk

The goal of this talk is threefold:

1. A first shot about linear independence of characters of enveloping algebras w.r.t. some algebras of nilpotents (Mathoverflow), extends to bialgebras (cocommutative or not), two proofs. This result is one of the three variations of a general theme [11].
2. Application to algebraic independence of some group of series w.r.t. polynomials (built on formal power series).
3. More on the structure Hausdorff groups: One-parameter groups, local system of coordinates, identities, motivations ...

Parts of this work are connected with Dyson series and take place within the project: *Evolution Equations in Combinatorics and Physics*.

Conclusion(s): More applications and perspectives.

Initial motivation (one of)

Lappo-Danilevskij's setting

J. A. Lappo-Danilevskij (J. A. Lappo-Danilevsky), Mémoires sur la théorie des systémes des équations différentielles linéaires. Vol. I, *Travaux Inst. Physico-Math. Stekloff*, 1934, Volume 6, 1–256

§ 2. HYPERLOGARITHMES 159

§ 2. Hyperlogarithmes. En abordant la résolution algorithmique du problème de Poincaré, nous introduisons le système des fonctions

$$L_b(a_{j_1}, a_{j_2}, \ldots, a_{j_\nu} | x), \qquad (j_1, j_2, \ldots, j_\nu = 1, 2, \ldots, m;\ \nu = 1, 2, 3 \ldots)$$

définies par les relations de récurrence:

$$(10) \quad \begin{aligned} L_b(a_{j_1} | x) &= \int_b^x \frac{dx}{x - a_{j_1}} = \log \frac{x - a_{j_1}}{b - a_{j_1}}; \\ L_b(a_{j_1} a_{j_2} \ldots a_{j_\nu} | x) &= \int_b^x \frac{L_b(a_{j_1} \ldots a_{j_{\nu-1}} | x)}{x - a_{j_\nu}}\, dx, \end{aligned}$$

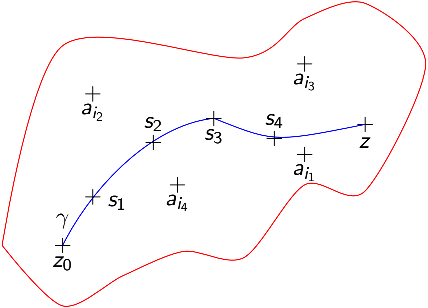
où b est un point fixe à distance finie, distinct des points $a_1, a_2, \ldots, a_m$. Ces fonctions seront nommées *hyperlogarithmes de la première espèce de*

Initial motivation (one of)/2

Let $(a_i)_{1\leq i\leq n}$ be a family of complex numbers (all different) and $z_0 \notin \{a_i\}_{1\leq i\leq n}$, then

Definition [Lappo-Danilevskij, 1928]

$$L(a_{i_1},\ldots,a_{i_n}|z_0 \overset{\gamma}{\rightsquigarrow} z) = \int_{z_0}^{z}\int_{z_0}^{s_n}\ldots\Big[\int_{z_0}^{s_1}\frac{ds}{s-a_{i_1}}\Big]\ldots\frac{ds_n}{s_n-a_{i_n}}.$$



Remarks

1. The result depends only on the homotopy class of the path and then the result is a holomorphic function on $\widetilde{B}$ $(B = \mathbb{C} \setminus \{a_1, \cdots, a_n\})$
2. From the fact that they are holomorphic, we can also study them in a section i.e. an open (simply connected) subset like the following cleft plane Ω

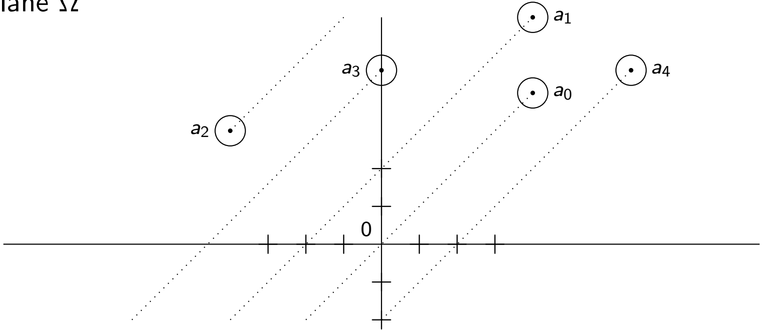


Figure: The cleft plane: one has an embedding $\mathcal{H}(\widetilde{B}) \hookrightarrow \mathcal{H}(\Omega)$.

Remarks/2

3. The set of functions

$$\alpha_{z_0}^{z}(\underbrace{x_{i_1}\ldots x_{i_n}}_{word}) = L(\underbrace{a_{i_1},\ldots,a_{i_n}}_{list}|z_0 \overset{\gamma}{\rightsquigarrow} z)$$

(or $1_{\mathcal{H}(B)}$ if the word is void) has a lot of nice combinatorial properties through its generating series

$$\sum_{w \in X} \alpha_{z_0}^{z}(w)\, w$$

- Noncommutative DE with left multiplier $\rightarrow$ Shuffle morphism
- Linear independence $\rightarrow$ to be extended to larger sets of scalars
- Factorisation $\rightarrow$ as characters
- Possiblity of left or right multiplicative renormalization at a neighbourhood of the singularities
- Extension to (some) series

Why do we code by words ?

In order to use the rich allowance of notations invented by algebraists, computer scientists, combinatorialists and physicists about NonCommutative Formal Power Series (NCFPS[a]), we have above coded the lists by words which will permit to do linear algebra and topology on the indexing.

$$(a_{i_1}, \ldots, a_{i_n}) \to w = x_{i_1} \ldots x_{i_n}$$

Note that Lappo-Danilevskij recursion is from left to right, we have used here right to left indexing to match with [6, 18, 21, 22] [b]

In particular, we will use the Kleene Star of series without constant term defined by

$$S^* = 1 + S + S^2 + \cdots = (1 - S)^{-1}$$

[a] See the body of knowledge developed in the series of conferences like SLC (Séminaire Lotharingien de Combinatoire) or FPSAC (Formal Power Series and Algebraic Combinatorics).

[b] Data structures are Letters in [6, 18] and Vector fields in [21].

Domain of $HL_\bullet$.

We now have an arrow of commutative algebras

$$(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \xrightarrow{HL_\bullet} (\mathcal{H}(\Omega), \times, 1_\Omega)$$

on the left $\mathbb{C}\langle X\rangle \hookrightarrow \mathbb{C}\langle\langle X\rangle\rangle$ is endowed with the Krull topology (coefficientwise stationary convergence) and, on the right $\mathcal{H}(\Omega)$ is endowed with the (Fréchet) topology of compact convergence.
We are led to the following definition.

Definition [Domain of $HL_\bullet$]

We define $Dom(HL_\bullet;\Omega)$ (or $Dom(HL_\bullet)$ if the context is clear) as the set of series $S = \sum_{n\geq 0} S_n$ (where $S_n = \sum_{|w|=n}\langle S|w\rangle\, w$, i.e. the decomposition is done by homogeneous slices) such that $\sum_{n\geq 0} HL_\bullet(S_n, z)$ converges unconditionally[a] for the compact convergence in Ω. One then sets $HL_\bullet(S, z) := \sum_{n\geq 0} HL_\bullet(S_n, z)$.

[a] In order to use functional properties of $\mathcal{H}(\Omega)$.

Domain of $HL_\bullet/2$

Diagram

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{HL_\bullet} & \mathbb{C}\{HL(w,z)\}_{w\in X^*} (= Span_{\mathbb{C}}\{HL(w,z)\}_{w\in X^*}) \\
\downarrow & & \downarrow \\
\mathbb{C}\langle\langle X\rangle\rangle \supset Dom(HL_\bullet) & \longrightarrow & \mathcal{H}(\Omega)
\end{array}$$

Proposition

With this definition, we have

1. $Dom(HL_\bullet)$ is a shuffle unital subalgebra[a] of $\mathbb{C}\langle\langle X\rangle\rangle$ and then so is $Dom^{rat}(HL_\bullet) := Dom(HL_\bullet) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$
2. For $S, T \in Dom(HL_\bullet)$, we have

$$HL_\bullet(S ш T) = HL_\bullet(S).HL_\bullet(T) \text{ and } HL_\bullet(1_{X^*}) = 1_{\mathcal{H}(\Omega)}$$

[a] This is due to the fact that $\mathcal{H}(\Omega)$ is nuclear, see [10].

Particular case: The ladder of polylogarithms

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Domain of $\mathrm{Li}_\bullet$ (particular case of $Dom(HL_\bullet)$)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S = \sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges for the compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z) \qquad (1)$$

Examples

$$Li_{x_0^*}(z) = z,\ Li_{x_1^*}(z) = (1-z)^{-1}\ ;\ Li_{(\alpha x_0+\beta x_1)^*}(z) = z^{\alpha}(1-z)^{-\beta}$$

Useful properties

Star of the plane property

Every conc-character is of the form $\left(\sum_{x\in X}\alpha(x)\,x\right)^*$

We will see that, with the common pattern (3 first examples)

$$w \sqcup\!\sqcup_\varphi 1_{X^*} = 1_{X^*} \sqcup\!\sqcup_\varphi w = w \text{ and}$$

$$au \sqcup\!\sqcup_\varphi bv = a(u \sqcup\!\sqcup_\varphi bv) + b(au \sqcup\!\sqcup_\varphi v) + \varphi(a,b)(u \sqcup\!\sqcup_\varphi v)$$

We get the following examples

Shuffle: $(\alpha x)^* \sqcup\!\sqcup (\beta y)^* = (\alpha x + \beta y)^*$ $(\varphi \equiv 0)$

Stuffle: $(\alpha y_i)^* \uplus (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$ $(\varphi(y_i, y_j) = y_{i+j})$

q-**infiltration**:
$(\alpha x)^* \uparrow_q (\beta y)^* = (\alpha x + \beta y + \alpha\beta q\delta_{x,y} x)^*$ $(\varphi(x,y) = q\delta_{x,y} x)$

Hadamard: $(\alpha a)^* \odot (\beta b)^* = 1_{X^*}$ if $a \neq b$ and $(\alpha a)^* \odot (\beta a)^* = (\alpha\beta a)^*$

Independence of characters w.r.t. polynomials.

mathoverflow

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Independence of characters with respect to polynomials

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6

2

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U}=\mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U}\to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+ = \ker(\epsilon)$. We build the following filtrations ($N\geq 0$)

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+\ldots\mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_{N+1} = \mathcal{U}.\mathcal{U}_N$) and, for $N\geq -1$

Independence of characters w.r.t. polynomials./2

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra. We note ϵ its counit and set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations ($N \geq 0$)

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots . \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_{N+1} = \mathcal{U}.\mathcal{U}_N$) and, for $N \geq -1$

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing (in particular $\mathcal{U}_{-1}^* = \{0\}$, $\mathcal{U}_0^* = k.\epsilon$).
One shows easily that, for $p, q \geq 0$ (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n \geq 0} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Independence of characters w.r.t. polynomials./3

Now, we can state the

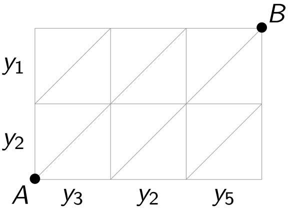
Theorem (From MO, k ring without zero divisors)

The set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$ is linearly free w.r.t. $\mathcal{U}^*_\infty$.

Remark

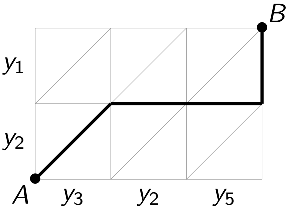
i) $\mathcal{U}^*_\infty$ *is a commutative k-algebra.*
ii) The title ("Independence of characters ...") comes from the fact that, with $(k\langle X\rangle, conc, 1)$ (non commutative polynomials), k a $\mathbb{Q}$-algebra (without zero divisors) and one of the usual comultiplications (with Δ_+ cocommutative and nilpotent, as co-shufflle, co-stuffle or - commutatively - deformed), if one takes $\mathfrak{g}$ as the space of primitive elements, we have $\mathcal{U}^* = k\langle\langle X\rangle\rangle$ (series) and $\mathcal{U}^*_\infty = k\langle X\rangle$.

With $Y = \{y_i\}_{i\geq 1}$, one can see the product $u \sqcup\!\sqcup_\varphi v$ as a sum indexed by paths (with right-up-diagonal steps) within the grid formed by the two words (u horizontal and v vertical, the diagonal steps corresponding to the factors $\varphi(a,b)$)



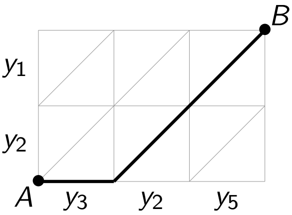
Computation of $y_2 y_1 \sqcup\!\sqcup_\varphi y_3 y_2 y_5$

For example, the path



evaluates as $\varphi(y_2, y_3) y_2 y_5 y_1$

the path



reads $y_3\varphi(y_2, y_2)\varphi(y_1, y_5)$.

We have the following

Theorem (Radford theorem for $\shuffle_\varphi$)

Let $\mathbf{k}$ *be a* $\mathbb{Q}$*-algebra (associative, commutative with unit) such that*

$$\shuffle_\varphi : \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle$$

is associative and commutative then

- $(\mathcal{L}yn(X)^{\shuffle_\varphi \alpha})_{\alpha\in\mathbb{N}^{(\mathcal{L}yn(X))}}$ *is a linear basis of* $\mathbf{k}\langle X\rangle$.
- *This entails that* $(\mathbf{k}\langle X\rangle, \shuffle_\varphi, 1_{X^*})$ *is a polynomial algebra with* $\mathcal{L}yn(X)$ *as transcendence basis.*

Making (combinatorial) bialgebras

Proposition

Let $\mathbf{k}$ *be a commutative ring (with unit). We suppose that the product* φ *is associative, then, on the algebra* $(\mathbf{k}\langle X\rangle, ш_\varphi, 1_{X^*})$*, we consider the comultiplication* Δ_{conc} *dual to the concatenation*

$$\Delta_{conc}(w) = \sum_{uv=w} u \otimes v \tag{2}$$

and the "constant term" character $\varepsilon(P) = \langle P|1_{X^*}\rangle$*.*
Then

(i) *With this setting, we have a bialgebra* [a]*.*

$$\mathcal{B}_\varphi = (\mathbf{k}\langle X\rangle, ш_\varphi, 1_{X^*}, \Delta_{conc}, \varepsilon) \tag{3}$$

(ii) *The bialgebra (eq. 3) is, in fact, a Hopf Algebra.*

[a] Commutative and, when $|X| \geq 2$, noncocommutative.

Dualizability

If one considers φ as defined by its structure constants

$$\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^{z}\,z$$

one sees at once that $ш_\varphi$ is dualizable within $\mathbf{k}\langle X\rangle$ iff the tensor $\gamma_{x,y}^{z}$ is locally finite in its contravariant place "z" i.e.

$$(\forall z\in X)(\#\{(x,y)\in X^2|\gamma_{x,y}^{z}\neq 0\}<+\infty)\ .$$

Remark

Shuffle, stuffle and infiltration are dualizable. The comultiplication associated with the stuffle with negative indices is not.

Dualizability/2

In the case when $\shuffle_\varphi$ is dualizable, one has a comultiplication

$$\Delta_{\shuffle_\varphi} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$$

such that, for all $u, v, w \in X^*$

$$\langle u \shuffle_\varphi v | w\rangle = \langle u \otimes v | \Delta_{\shuffle_\varphi}(w)\rangle \tag{4}$$

Then, the following

$$\mathcal{B}_\varphi^\vee = (\mathbf{k}\langle X\rangle, conc, 1_{X^*}, \Delta_{\shuffle_\varphi}, \varepsilon) \tag{5}$$

is a bialgebra in duality with $\mathcal{B}_\varphi$ (not always a Hopf algebra although $\mathcal{B}$ was so, for example, see $\mathcal{B}$ with $\shuffle_\varphi = \uparrow_q$ i.e. the q-infiltration).

The interest of these bialgebras is that they provide a host of easy-to-within-compute bialgebras with easy-to-implement-and-compute set of characters through the following proposition.

Proposition (Conc-Bialgebras)

Let $\mathbf{k}$ *be a commutative ring,* X *a set and* $\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^z\, z$ *an associative and dualizable law on* $\mathbf{k}.X$. *Let* $\shuffle_\varphi$ *and* $\Delta_{\shuffle_\varphi}$ *be the associated product and co-product. Then:*
i) $\mathcal{B}=(\mathbf{k}\langle X\rangle, conc, 1_{X^*}, \Delta_{\shuffle_\varphi}, \epsilon)$ *is a bialgebra which, in case* $\mathbb{Q}\hookrightarrow\mathbf{k}$, *is an enveloping algebra iff* φ *is commutative and* $\Delta_{\shuffle_\varphi}^+$ *nilpotent.*
ii) In the general case $S\in\mathbf{k}\langle\langle X\rangle\rangle=\mathbf{k}\langle X\rangle^\vee$ *is a character for* $\mathcal{A}=(\mathbf{k}\langle X\rangle, conc, 1_{X^*})$ *(i.e. a conc-character) iff it is of the form*

$$S=(\sum_{x\in X}\alpha_x\, x)^*=\sum_{n\geq 0}(\sum_{x\in X}\alpha_x\, x)^n \text{ and, with this notation} \tag{6}$$

$$(\sum_{x\in X}\alpha_x\, x)^* \shuffle_\varphi (\sum_{x\in X}\beta_y\, y)^* = \left(\sum_{z\in X}(\alpha_z+\beta_z)\, z+\sum_{x,y\in X}\alpha_x\beta_y\,\varphi(x,y)\right)^* \tag{7}$$

GD, Darij Grinberg and Hoang Ngoc Minh *Three variations on the linear independence of grouplikes in a coalgebra*, [arXiv:2009.10970]

GD, Quoc Huan Ngô and V. Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

Main result about independence of characters w.r.t.

Theorem (G.D., Darij Grinberg, H. N. Minh)

Let $\mathcal{B}$ be a $\mathbf{k}$*-bialgebra. As usual, let $\Delta = \Delta_{\mathcal{B}}$ and $\epsilon = \epsilon_{\mathcal{B}}$ be its comultiplication and its counit.*
Let $\mathcal{B}_+ = \ker(\epsilon)$. For each $N \geq 0$, let $\mathcal{B}_+^N = \underbrace{\mathcal{B}_+ \cdot \mathcal{B}_+ \cdot \cdots \cdot \mathcal{B}_+}_{N \text{ times}}$, where $\mathcal{B}_+^0 = \mathcal{B}$. Note that $\left(\mathcal{B}_+^0, \mathcal{B}_+^1, \mathcal{B}_+^2, \ldots\right)$ is called the <u>*standard decreasing filtration*</u> *of $\mathcal{B}$.*
For each $N \geq -1$, we define a $\mathbf{k}$*-submodule $\mathcal{B}_N^\vee$ of $\mathcal{B}^\vee$ by*

$$\mathcal{B}_N^\vee = (\mathcal{B}_+^{N+1})^\perp = \left\{ f \in \mathcal{B}^\vee \mid f\left(\mathcal{B}_+^{N+1}\right) = 0 \right\}. \tag{8}$$

Thus, $\left(\mathcal{B}_{-1}^\vee, \mathcal{B}_0^\vee, \mathcal{B}_1^\vee, \ldots\right)$ is an increasing filtration of $\mathcal{B}_\infty^\vee := \bigcup_{N \geq -1} \mathcal{B}_N^\vee$ with $\mathcal{B}_{-1}^\vee = 0$.

Theorem (DGM, cont'd)

Let also $\Xi(\mathcal{B})$ be the monoid (group, if $\mathcal{B}$ is a Hopf algebra) of characters of the algebra $(\mathcal{B}, \mu_{\mathcal{B}}, 1_{\mathcal{B}})$.
Then:

(a) *We have $\mathcal{B}_p^\vee \circledast \mathcal{B}_q^\vee \subseteq \mathcal{B}_{p+q}^\vee$ for any $p, q \geq -1$ (where we set $\mathcal{B}_{-2}^\vee = 0$). Hence, $\mathcal{B}_\infty^\vee$ is a subalgebra of the convolution algebra $\mathcal{B}^\vee$.*

(b) *Assume that $\mathbf{k}$ is an integral domain. Then, the set $\Xi(\mathcal{B})^\times$ of invertible characters (i.e., of invertible elements of the monoid $\Xi(\mathcal{B})$) is left $\mathcal{B}_\infty^\vee$-linearly independent.*

Remark

The standard decreasing filtration of $\mathcal{B}$ is weakly decreasing, it can be stationary after the first step. An example can be obtained by taking the universal enveloping bialgebra of any simple Lie algebra (or, more generally, of any perfect Lie algebra); it will satisfy $\bigcap_{n\geq 0} \mathcal{B}_+^n = \mathcal{B}_+$.

Corollary

We suppose that $\mathcal{B}$ is cocommutative, and $\mathbf{k}$ is an integral domain. Let $(g_x)_{x \in X}$ be a family of elements of $\Xi(\mathcal{B})^\times$ (the set of invertible characters of $\mathcal{B}$), and let $\varphi_X : \mathbf{k}[X] \to (\mathcal{B}^\vee, \circledast, \epsilon)$ be the $\mathbf{k}$-algebra morphism that sends each $x \in X$ to g_x. In order for the family $(g_x)_{x \in X}$ (of elements of the commutative ring $(\mathcal{B}^\vee, \circledast, \epsilon)$) to be algebraically independent over the subring $(\mathcal{B}^\vee_\infty, \circledast, \epsilon)$, it is necessary and sufficient that the monomial map

$$\begin{aligned} m : \mathbb{N}^{(X)} &\to (\mathcal{B}^\vee, \circledast, \epsilon), \\ \alpha &\mapsto \varphi_X(X^\alpha) = \prod_{x \in X} g_x^{\alpha_x} \end{aligned} \tag{9}$$

(where α_x means the x-th entry of α) be injective.

Examples

Let $\mathbf{k}$ be an integral domain, and let us consider the standard bialgebra $\mathcal{B} = (\mathbf{k}[x], \Delta, \epsilon)$ For every $c \in \mathbf{k}$, there exists only one character of $\mathbf{k}[x]$ sending x to c; we will denote this character by $(c.x)^* \in \mathbf{k}[[x]]$ (motivation about this notation is Kleene star). Thus, $\Xi(\mathcal{B}) = \{(c.x)^* \mid c \in \mathbf{k}\}$. It is easy to check that $(c_1.x)^* \sqcup\!\sqcup (c_1.x)^* = \left((c_1 + c_2).x\right)^*$ for any $c_i \in \mathbf{k}$ (‡). Thus, any $c_1, c_2, \ldots, c_k \in \mathbf{k}$ and any $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ satisfy

$$\left((c_1.x)^*\right)^{\sqcup\!\sqcup\, \alpha_1} \sqcup\!\sqcup \left((c_2.x)^*\right)^{\sqcup\!\sqcup\, \alpha_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \left((c_k.x)^*\right)^{\sqcup\!\sqcup\, \alpha_k} = \left((\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k).x\right)^* . \tag{10}$$

From (‡) above, the monoid $\Xi(\mathcal{B})$ is isomorphic with the abelian group $(\mathbf{k}, +, 0)$; in particular, it is a group, so that $\Xi(\mathcal{B})^\times = \Xi(\mathcal{B})$.

Examples/2

Take $\mathbf{k} = \overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$) and $c_n = \sqrt{p_n} \in \mathbf{k}$, where p_n is the n-th prime number. What precedes shows that the family of series $\big((\sqrt{p_n}x)^*\big)_{n\geq 1}$ is algebraically independent over the polynomials (i.e., over $\overline{\mathbb{Q}}[x]$) within the commutative $\overline{\mathbb{Q}}$-algebra $\big(\overline{\mathbb{Q}}[[x]], ш, 1\big)$. This example can be double-checked using partial fractions decompositions as, in fact, $(\sqrt{p_n}x)^* = \dfrac{1}{1-\sqrt{p_n}x}$ (this time, the inverse is taken within the ordinary product in $\mathbf{k}[[x]]$) and

$$\left(\frac{1}{1-\sqrt{p_n}x}\right)^{ш\, n} = \frac{1}{1-n\sqrt{p_n}x}.$$

Examples/3

All what has been said holds true for characters with values within some $\mathcal{A} \in$ **k-CAAU**. For example, let us consider the polylogarithms and $\mathcal{A} = \mathbb{C}[z^{\alpha}(1-z)^{-\beta}]_{\alpha,\beta\in\mathbb{R}} = \mathcal{C}_{\mathbb{R}}$.
We have to suppose $P_i, i = 1 \ldots 3$ in $\mathcal{A}$ such that

$$P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_{\Omega}$$

and then, either pass to meromorphic functions or use the localized BTT.

Proof that $[1_\Omega, \log(z), \log(\frac{1}{1-z})]$ is $\mathcal{C}_\mathbb{R}$-free.

We first prove that $P_2 = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}$ is zero using the deck transformation D_0 of index one around zero.
One has $D_0^n(\sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}) = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i} e^{2i\pi.n\alpha_i}$, the same calculation holds for all P_i which proves that all $D_0^n(P_i)$ are bounded. But one has $D_0^n(\log(z)) = \log(z) + 2i\pi.n$ and then

$$D_0^n(P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z}))) =$$

$$D_0^n(P_1(z)) + D_0^n(P_2(z))(\log(z) + 2i\pi.n) + D_0^n(P_3(z))\log(\frac{1}{1-z}) = 0$$

It suffices to build a sequence of integers $n_j \to +\infty$ such that $\lim_{j\to\infty} D_0^{n_j}(P_2(z)) = P_2(z)$ which is a consequence of the following lemma.

Lemma

Let us consider a homomorphism $\varphi : \mathbb{N} \to G$ *where* G *is a compact (Hausdorff) group, then it exists* $u_j \to +\infty$ *such that*

$$\lim_{j\to\infty} \varphi(u_j) = e$$

Proof.

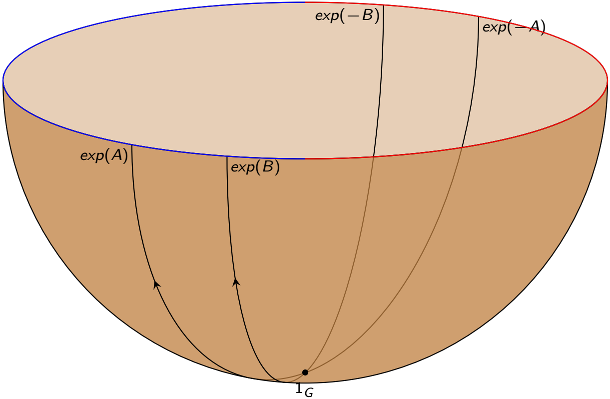
First of all, due to the compactness of G, the sequence $\varphi(n)$ admits a subsequence $\varphi(n_k)$ convergent to some $\ell \in G$. Now one can refine the sequence as n_{k_j} such that

$$0 < n_{k_1} - n_{k_0} < \ldots < n_{k_{j+1}} - n_{k_j} < n_{k_{j+2}} - n_{k_{j+1}} < \ldots$$

With $u_j = n_{k_{j+1}} - n_{k_j}$ one has $\lim_{j\to\infty} \varphi(u_j) = e$.
End of the proof One applies the lemma to the morphism

$$n \mapsto (e^{2i\pi.n\alpha_i})_{i\in F} \in \mathbb{U}^F$$

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{\text{Ш}}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

Hausdorff group of the stuffle Hopf algebra.

With $Y = \{y_i\}_{i\geq 1}$ and

$$\Delta_{\uplus}(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j$$

the bialgebra $\mathcal{B} = (\mathbf{k}\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{\uplus}, \epsilon)$ is an enveloping algebra (it is cocommutative, connex and graded by the weight function given by $||y_{i_1}y_{i_2}\cdots y_{i_k}|| = \sum_{s=1}^{k} i_s$ on a word $w = y_{i_1}y_{i_2}\cdots y_{i_k}$).
With $\varphi(y_i, y_j) = y_{i+j}$, (eq.7) gives

$$(\sum_{i\geq 1} \alpha_i\, y_i)^*_{\uplus}(\sum_{j\geq 1} \beta_j\, y_j)^* = (\sum_{i\geq 1} \alpha_i\, y_i + \sum_{j\geq 1} \beta_j\, y_j + \sum_{i,j\geq 1} \alpha_i\beta_j\, y_{i+j})^* \tag{11}$$

This formula suggests us to code, in an umbral style, $\sum_{k\geq 1} \alpha_k\, y_k$ by the series $\sum_{k\geq 1} \alpha_k\, x^k \in \mathbf{k}_+[[x]]$. Indeed, we get the following proposition whose first part, characteristic-freely describes the group of characters $\Xi(\mathcal{B})$ and its law and the second part, about the exp-log correspondence, requires $\mathbf{k}$ to be $\mathbb{Q}$-algebra.

Proposition

Let π_Y^{Umbra} *be the linear isomorphism* $\mathbf{k}_+[[x]] \to \widehat{\mathbf{k}.Y}$ *defined by*

$$\sum_{n\geq 1} \alpha_n\, x^n \mapsto \sum_{k\geq 1} \alpha_k\, y_k \qquad (12)$$

Then

1. *One has, for* $S, T \in \mathbf{k}_+[[x]]$*,*

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^* \qquad (13)$$

2. *From now on* $\mathbf{k}$ *is supposed to be a* $\mathbb{Q}$*-algebra.*
For $t \in \mathbf{k}$ *and* $T \in \mathbf{k}_+[[x]]$*, the family* $(\frac{(t.T)^n}{n!})_{n\geq 0}$ *is summable and one sets*

$$G(t) = (\pi_Y^{Umbra}(e^{t.T} - 1))^* \qquad (14)$$

Proposition (Cont'd)

3. *The parametric character G fulfills the "stuffle one-parameter group" property i.e. for $t_1, t_2 \in \mathbf{k}$, we have*

$$G(t_1 + t_2) = G(t_1) \uplus G(t_2) \ ; \ G(0) = 1_{Y^*} \tag{15}$$

4. *We have*

$$G(t) = exp_{\uplus}\left(t.\pi_Y^{Umbra}(T)\right) \tag{16}$$

5. *In particular, calling π_x^{Umbra} the inverse of π_Y^{Umbra} we get, for $P^* \in \Xi(\mathcal{B})$ (in other words $P \in \widehat{\mathbf{k}.Y}$),*

$$\log_{\uplus}(P^*) = \pi_Y^{Umbra}(\log(1 + \pi_x^{Umbra}(P))) \tag{17}$$

Proof (Sketch)

i) We have

$$\pi_Y^{Umbra}(S) = \sum_{i \geq 1} \langle S | x^i \rangle y_i \quad \pi_Y^{Umbra}(T) = \sum_{j \geq 1} \langle T | x^j \rangle y_j$$

and then

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\sum_{i \geq 1} \langle S | x^i \rangle y_i)^* \uplus (\sum_{j \geq 1} \langle T | x^j \rangle y_j) =$$
$$\Big(\sum_{i \geq 1} \langle S | x^i \rangle y_i) + \sum_{j \geq 1} \langle T | x^j \rangle y_j + \sum_{i,j \geq 1} \langle S | x^i \rangle \langle T | x^j \rangle y_{i+j} \Big)^* =$$
$$(\pi_Y^{Umbra}(S + T + ST))^* = (\pi_Y^{Umbra}((1+S)(1+T) - 1))^*$$

ii.1) The one parameter group property is a consequence of (13) applied to the series $e^{t_i . T} - 1, \; i = 1, 2$.

Proof (Sketch)/2

ii.2) Property 15 holds for every $\mathbb{Q}$-algebra, in particular in $\mathbf{k}_1 = \mathbf{k}[t]$ and $\mathbf{k}_1\langle\langle Y\rangle\rangle$ is endowed with the structure of a differential ring by term-by-term derivations (see [8] for formal details). We can write $G(t) = 1 + t.G_1 + t^2.G_2(t)$ (where $G_1 = \pi_Y^{Umbra}(T)$ is independent from t) and from 15, we have

$$G'(t) = G_1.G(t)\ ;\ \ G(0) = 1_{Y^*} \tag{18}$$

but $H(t) = exp_{\sqcup\!\!\sqcup}(t.G_1)$ satisfies 18 whence the equality.
ii.3) At $t = 1$, we have $\exp_{\sqcup\!\!\sqcup}(\pi_Y^{Umbra}(T)) = (\pi_Y^{Umbra}(e^T - 1))^*$ hence, with $P = \pi_Y^{Umbra}(e^T - 1)$ (take $T := \log(\pi_x^{Umbra}(P) + 1)$)

$$\pi_Y^{Umbra}(T) = \log_{\sqcup\!\!\sqcup}(P^*) \qquad \text{[QED]} \tag{19}$$

Application of (17)

$$(ty_k)^* = \exp_{\sqcup\!\!\sqcup}\Big(\sum_{n\geq 1}\frac{(-1)^{n-1}t^n y_{nk}}{n}\Big) \tag{20}$$

Conclusion(s): More applications and perspectives.

1. Star of the plane property (slide 17) holds for non-commutative valued (as matrix-valued) characters.
2. Combinatorial study of other $\shuffle_{\varphi}$ one-parameter groups and evolution equations in convolution algebras.
3. Factorisation of $\mathcal{A}$-valued characters ($\mathcal{A}$ **k**-CAAU).
   For example, with

$$\mathcal{B} = (\mathbf{k}\langle X\rangle, \shuffle, 1_{X^*}, \Delta_{conc}, \epsilon),\ \mathcal{A} = (\mathbf{k}\langle X\rangle, \shuffle, 1_{X^*}),\ \chi = Id$$

(χ is a shuffle character) one has (MRS factorisation)

$$\Gamma(\chi) = \sum_{w\in X^*} Id(w) \otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \qquad (21)$$

MRS : (Mélançon, Reutenauer, Schützenberger)

Conclusion(s): More applications and perspectives./2

4. Deformed version of factorisation above for $\sqcup\!\sqcup_\varphi$ (with φ associative, commutative, dualisable and moderate). With

$$\mathcal{B} = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup_\varphi, 1_{X^*}, \Delta_{conc}, \epsilon)\,,\ \mathcal{A} = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup_\varphi, 1_{X^*})\,,\ \chi = Id$$

(χ is a shuffle character) one has

$$\Gamma(\chi) = \sum_{w\in X^*} Id(w)\otimes w = \sum_{w\in X^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(\Sigma_l \otimes \Pi_l) \tag{22}$$

5. Holds for all enveloping algebras which are free as **k**-modules (with $\mathbb{Q} \to \mathbf{k}$). This could help to the combinatorial study of the group of characters of enveloping algebras of Lie algebras like DK[a]-Lie algebras and other ones, or deformed.

[a] Drinfeld–Kohno.

Thank you for your attention.

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